Journal of Engineering Mathematics 18 (1984) 37–62

The solution of a mixed boundary value problem in the theory of diffraction

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(Received September 24, 1979 and in final form September 7, 1983)

Summary

An exact solution is obtained for the problem of the diffraction of a cylindrical sound wave by an absorbent semi-infinite plane. The two faces of the half-plane have different impedance boundary conditions. The problem which is solved is a mathematical model for a noise barrier whose surface is treated with two different acoustically absorbent materials.

The usual Wiener-Hopf method (which is the standard technique for solving half-plane problems) has to be modified to give a solution to the present mixed boundary value problem.

1. Introduction

Unwanted noise from motorways, railways and airports can be shielded by a barrier which intercepts the line-of-sight from the noise source to a receiver. The acoustic field in the shadow region of a barrier, when transmission through the barrier is negligible, is due to the diffraction at the edge alone. The design and performance of noise barriers, particularly for the reduction of traffic noise, has received considerable attention in recent years; see the review article by Kurze [1]. Noise shielding by barriers (aircraft wings) also has important applications in aircraft noise reduction; see the review article by Jones [2].

An effective way of reducing the noise level in the shadow region of a barrier is to line one or both faces of the barrier with absorbent material. The rationale for such a noise barrier design is given in Rawlins [3]. The presence of an acoustically absorbing lining on a surface is described by an impedance relationship between the acoustic pressure (p) and the normal acoustic velocity fluctuation on the lining surface (Morse and Ingard [4]). This gives rise to a boundary condition on the absorbent lining of the form

$$\frac{\partial p}{\partial n} = ik\beta p, \qquad \operatorname{Re}\beta > 0, \tag{$*$}$$

where the sound wave has time harmonic variation $e^{-i\omega t}$, and $k = \omega/c$; c is the velocity of sound, $\boldsymbol{n}$ the normal pointing into the lining, and β the complex specific admittance of the acoustic lining. An acoustically hard (or perfectly reflecting) surface has a vanishing admittance, i.e. $|\beta| \to 0$, and an acoustically soft surface (pressure fluctuation vanishing on surface) is given by $|\beta| \to \infty$.

If the wavelength of the sound is much smaller than the length scale associated with the barrier, the diffraction process is governed only by the local conditions at the edge. Hence a rigid noise barrier with absorbent material on one face can be modelled by a semi-infinite plane, one face of which is absorbent and the other rigid. The solution of this problem for the special case $|\beta| \to \infty$ (i.e. the absorbent surface boundary condition $(*)$ replaced by a soft boundary condition) was given by Rawlins [3]. This problem corresponds to the physical situation of diffraction by a half-plane, one face of which is acoustically soft and the other face being rigid. A modification of the standard Wiener-Hopf technique was used to obtain a solution of the problem. Later, Williams [5] obtained the same solution by a much simpler approach. However, the approach of [3] can be adapted to deal with the more complicated situation where β is finite. A similar approach to that used in Rawlins [3] and here, was followed by Hurd and Przeździecki [6,7] in their solution of the problem of plane-wave diffraction by a half-plane with different face impedances. However, it is shown that the present approach is more straightforward in that it separates the function-theoretic Wiener-Hopf factorisation of a matrix, from the boundary value problem analysis. This is the traditional approach for Wiener-Hopf problems. We shall show that the usual Jones' method [8] can be used to set up a system of Wiener-Hopf equations. These equations can be uncoupled if a matrix function can be factorised. It is shown that this is indeed the case, the factorisation being reduced to the solution of two standard Hilbert problems.

In Section 2 a boundary value problem is formulated for the diffraction of a cylindrical sound wave by a half-plane with different face impedances. To ensure a unique solution an "edge condition" (Jones [9]) is imposed. This edge condition is the usual one associated with diffraction theory (i.e. that the sound energy is bounded in a finite region around the edge of the half-plane).

In Section 3 a solution is obtained for the boundary value problem set up in Section 2. The method used is the standard Jones' technique of representing the acoustic potential function as a Fourier transform. This leads to a coupled system of Wiener-Hopf equations. To uncouple the equations, and therefore to be able to apply the usual Wiener-Hopf argument, a matrix function has to be factorised. This is carried out in Appendix A with the help of results given in Appendix B. In Section 4 the solution, which is in terms of double integrals, is asymptotically evaluated for source and receiver positions well removed from the edge of the half-plane. Explicit expressions are obtained for the diffracted field and the geometrical acoustic field.

2. Formulation of the boundary value problem

We shall consider small amplitude sound waves diffracted by a half-plane. The half-plane is assumed to occupy $x \leqslant 0$, $y = 0$, and to be infinitely thin and rigid with its surface treated with acoustically absorbent material (see Fig. 1). The upper surface ($x \leqslant 0, y = 0^+$) will therefore require the satisfaction of the absorbing boundary condition $p - Z_1 u_n = 0$, while on the lower surface ($x < 0, y = 0^-$) the boundary condition $p + Z_2 u_n = 0$ applies. Here p is the acoustic pressure and u_n is the normal component of the perturbation velocity at a point on the surface of the half-plane. The acoustic impedance of the upper (lower) surface is $Z_1(Z_2)$. We shall restrict our consideration to a harmonic time dependence, with the time factor $e^{-i\omega t}$ being suppressed throughout.

The perturbation velocity $\boldsymbol{u}$ of the irrotational sound waves can be expressed in terms of the velocity potential $\chi(x, y)$ by $\boldsymbol{u} = \text{grad}\ \chi$. The resulting pressure in the sound field is given by $p = i\omega\rho_0\chi(x, y)$ where ρ_0 is the density of the initially undisturbed ambient medium.

The primary source is taken to be a line source, parallel to the half-plane edge, at a position (x_0, y_0), $y_0 > 0$. The problem we are considering becomes one of solving the wave equation

$$\frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial y^2} + k^2 \chi = \delta(x - x_0)\delta(y - y_0), \tag{1}$$

in all space excluding the half-plane; here $k = \omega/c$ and c is the speed of sound in the initially undisturbed medium. The effect of the half-plane is described by the boundary conditions

$$\left(\frac{\partial}{\partial y} + ik\beta_1\right)\chi(x, 0^+) = 0, \qquad x < 0, \tag{2}$$

$$\left(\frac{\partial}{\partial y} - ik\beta_2\right)\chi(x, 0^-) = 0, \qquad x < 0, \tag{3}$$

where $\beta_1 = \rho_0 c/Z_1$, $\beta_2 = \rho_0 c/Z_2$ are the specific admittances of the absorbent surfaces, and for acoustic absorption $\mathrm{Re}(\beta_1) > 0$, $\mathrm{Re}(\beta_2) > 0$, see Morse and Ingard [4].

In order that the solution to the boundary value problem (1–3) be unique, we shall also require that the field be continuous and that the edge shall not radiate any energy; also that the field should be radiating outwards at infinity, see Jones [9]. The condition that the edge does not behave like a source, and therefore radiate energy, requires that the field near the edge behave like

$$\chi = O(1), \qquad \mathrm{grad}\,\chi = O(r^{-1/2}), \tag{4}$$

$$\text{as } r = (x^2 + y^2)^{1/2} \to 0, \qquad |\beta_1| < \infty, |\beta_2| < \infty.$$

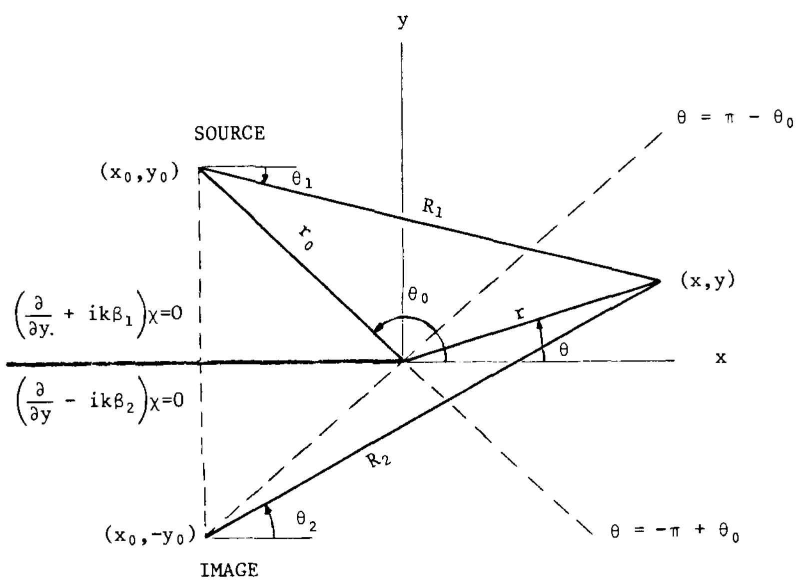

Figure 1. Geometry of the diffraction problem.

The behaviour of the edge field as given in the above expression is different to that given in Rawlins [3] where $|\beta_1| \to \infty$, $\beta_2 = 0$. We have here excluded the latter case and also $|\beta_2| \to \infty$, $\beta_1 = 0$. The reason being that the solution obtained is not uniformly continuous in the limit $|\beta_1| \to \infty$ or $|\beta_2| \to \infty$.

3. Solution of the boundary value problem

We shall assume, for analytical convenience, that $k = k_r + ik_i$, $k_r > 0$, $k_i \geqslant 0$. At the end of the analysis we can set $k_i = 0$.

Define $\hat{\chi}(\alpha, y)$, where α is a complex variable, by

$$\hat{\chi}(\alpha, y) = \int_{-\infty}^{\infty} \chi(x, y)\, e^{i\alpha x} dx. \tag{5}$$

The radiation condition requires that the phase dependence of $\chi(x, y)$, as $|x| \to \infty$, behave like $e^{-k_i|x|}$. In view of this it can be seen that $\hat{\chi}(\alpha, y)$ will exist for $-k_i < \text{Im}(\alpha) < k_i$. Then it follows from (1) that $\hat{\chi}(\alpha, y)$ satisfies

$$\frac{d^2\hat{\chi}}{dy^2} + \kappa^2\hat{\chi} = e^{i\alpha x_0}\delta(y - y_0), \qquad y_0 > 0, \tag{6}$$

where $\kappa = (k^2 - \alpha^2)^{1/2}$ is defined to be that branch for which $\kappa = k$ when $\alpha = 0$. Then κ will always have a positive imaginary part in the region $|\text{Im}(\alpha)| < k_i$. A solution of (6) for α in the strip $|\text{Im}(\alpha)| < k_i$, which decays as $|y| \to \infty$, is given by

$$\hat{\chi}(\alpha, y) = A(\alpha) \exp[i\kappa y] + \exp[i\{\alpha x_0 + \kappa|y - y_0|\}]/(2i\kappa), \qquad (y > 0), \tag{7}$$

$$= B(\alpha) \exp[-i\kappa y], \qquad (y < 0). \tag{8}$$

Let

$$\Phi_1^-(\alpha) = \int_{-\infty}^{0} [\chi(x, 0^+) - \chi(x, 0^-)]\, e^{i\alpha x} dx, \tag{9}$$

$$\Phi_2^-(\alpha) = \int_{-\infty}^{0} \left[\frac{\partial\chi}{\partial y}(x, 0^+) - \frac{\partial\chi}{\partial y}(x, 0^-)\right] e^{i\alpha x} dx, \tag{10}$$

$$\Psi_1^+(\alpha) = \int_0^{\infty} \left[\left(\frac{\partial}{\partial y} + ik\beta_1\right)\chi(x, 0^+)\right] e^{i\alpha x} dx, \tag{11}$$

$$\Psi_2^+(\alpha) = \int_0^{\infty} \left[\left(\frac{\partial}{\partial y} - ik\beta_2\right)\chi(x, 0^-)\right] e^{i\alpha x} dx. \tag{12}$$

Then $\Phi_{1,2}^-(\alpha)$ are analytic for $\text{Im}(\alpha) < k_i$, and $\Psi_{1,2}^+(\alpha)$ are analytic for $\text{Im}(\alpha) > -k_i$. Throughout this work a superscript (or subscript) plus or minus attached to any function will denote that the function is analytic in $\mathrm{Im}(\alpha) > -k_i$ or $\mathrm{Im}(\alpha) < k_i$, respectively. Using the expressions (2), (3), (5), (7) and (8) in the expressions (9) to (12) gives

$$\Phi_1^-(\alpha) = A(\alpha) - B(\alpha) + \exp[i\{\alpha x_0 + \kappa y_0\}]/(2i\kappa), \tag{13}$$

$$\Phi_2^-(\alpha) = i\kappa(A(\alpha) + B(\alpha)) - \exp[i\{\alpha x_0 + \kappa y_0\}]/2, \tag{14}$$

$$\Psi_1^+(\alpha) = A(\alpha)(i\kappa + ik\beta_1) + (-i\kappa + ik\beta_1)\exp[i\{\alpha x_0 + \kappa y_0\}]/(2i\kappa), \tag{15}$$

$$\Psi_2^+(\alpha) = -B(\alpha)(i\kappa + ik\beta_2). \tag{16}$$

Eliminating $A(\alpha)$ and $B(\alpha)$ from (13) to (16) gives the matrix Wiener-Hopf equation

$$\boldsymbol{\Psi}_+(\alpha) = \boldsymbol{K}(\alpha)\boldsymbol{\Phi}_-(\alpha) + \boldsymbol{D}(\alpha), \tag{17}$$

where

$$\boldsymbol{\Psi}_+(\alpha) = \begin{pmatrix} \Psi_1^+(\alpha) \\ \Psi_2^+(\alpha) \end{pmatrix}, \qquad \boldsymbol{\Phi}_-(\alpha) = \begin{pmatrix} \Phi_1^-(\alpha) \\ \Phi_2^-(\alpha) \end{pmatrix}, \tag{18}$$

$$\boldsymbol{K}(\alpha) = \tfrac{1}{2}\begin{pmatrix} i(\kappa + k\beta_1) & (\kappa + k\beta_1)/\kappa \\ i(\kappa + k\beta_2) & -(\kappa + k\beta_2)/\kappa \end{pmatrix}, \tag{19}$$

$$\boldsymbol{D}(\alpha) = \begin{pmatrix} (k\beta_1 - \kappa)\exp[i\{\alpha x_0 + \kappa y_0\}]/(2\kappa) \\ -(k\beta_2 + \kappa)\exp[i\{\alpha x_0 + \kappa y_0\}]/(2\kappa) \end{pmatrix}. \tag{20}$$

The expression (17) constitutes a coupled system of Wiener-Hopf equations. The standard Wiener-Hopf technique can only be applied if the system (17) can be uncoupled into two separate Wiener-Hopf equations. This requires that the matrix function $\boldsymbol{K}(\alpha)$ can be factorized. This is not a trivial operation and it is not always obvious that one can in fact factorise the matrix. In the present problem it is shown, in Appendix A, that the matrix $\boldsymbol{K}(\alpha)$ can be factorised such that

$$\boldsymbol{K}(\alpha) = \boldsymbol{U}(\alpha)\boldsymbol{L}^{-1}(\alpha), \tag{21}$$

where

$$\boldsymbol{U}(\alpha) = \begin{pmatrix} u_{11}(\alpha) & u_{12}(\alpha) \\ u_{21}(\alpha) & u_{22}(\alpha) \end{pmatrix}, \qquad \boldsymbol{L}(\alpha) = \begin{pmatrix} l_{11}(\alpha) & l_{12}(\alpha) \\ l_{21}(\alpha) & l_{22}(\alpha) \end{pmatrix}. \tag{22}$$

The elements of $\boldsymbol{U}(\alpha)$ are given, see Appendix A, by

$$u_{11}(\alpha) = -\left[\frac{\left(\sqrt{k+\alpha} + \sqrt{kB_1(+)}\right)\left(\sqrt{k+\alpha} + \sqrt{kB_1(-)}\right)}{\left(\sqrt{k+\alpha} + \sqrt{kB_2(+)}\right)\left(\sqrt{k+\alpha} + \sqrt{kB_2(-)}\right)}\right]^{1/2} \exp\left[\tfrac{1}{2}\int_\infty^\alpha Q(u)\,\mathrm{d}u\right], \tag{23}$$

$$u_{21}(\alpha)=\left[\frac{\left(\sqrt{k+\alpha}+\sqrt{kB_2(+)}\right)\left(\sqrt{k+\alpha}+\sqrt{kB_2(-)}\right)}{\left(\sqrt{k+\alpha}+\sqrt{kB_1(+)}\right)\left(\sqrt{k+\alpha}+\sqrt{kB_1(-)}\right)}\right]^{1/2}\exp\left[\tfrac{1}{2}\int_{\infty}^{\alpha}Q(u)\mathrm{d}u\right], \quad (24)$$

$$u_{12}(\alpha)=(k+\alpha)^{1/2}u_{11}(\alpha), \quad (25)$$

$$u_{22}(\alpha)=-(k+\alpha)^{1/2}u_{21}(\alpha), \quad (26)$$

where

$$Q(u)=\frac{-1}{u+k}+\frac{\cos^{-1}\left(-\sqrt{1-\beta_1^2}\right)}{2\pi\left(u+k\sqrt{1-\beta_1^2}\right)}+\frac{\cos^{-1}\left(\sqrt{1-\beta_1^2}\right)}{2\pi\left(u-k\sqrt{1-\beta_1^2}\right)}$$

$$+\frac{\cos^{-1}\left(-\sqrt{1-\beta_2^2}\right)}{2\pi\left(u+k\sqrt{1-\beta_2^2}\right)}+\frac{\cos^{-1}\left(\sqrt{1-\beta_2^2}\right)}{2\pi\left(u-k\sqrt{1-\beta_2^2}\right)}$$

$$-\frac{k\beta_1}{2\pi}\frac{\cos^{-1}(u/k)}{\sqrt{k^2-u^2}}\left\{\frac{1}{u+k\sqrt{1-\beta_1^2}}+\frac{1}{u-k\sqrt{1-\beta_1^2}}\right\}$$

$$-\frac{k\beta_2}{2\pi}\frac{\cos^{-1}(u/k)}{\sqrt{k^2-u^2}}\left\{\frac{1}{u+k\sqrt{1-\beta_2^2}}+\frac{1}{u-k\sqrt{1-\beta_2^2}}\right\}, \quad (27)$$

and

$$B_{1,2}(\pm)=1\pm\sqrt{1-\beta_{1,2}^2}. \quad (28)$$

The elements of $\boldsymbol{U}(\alpha)$: u_{ij}, $i, j=1, 2$, are analytic in $\mathrm{Im}(\alpha)>-k_i$. The elements of $\boldsymbol{L}(\alpha)$ are analytic in $\mathrm{Im}(\alpha)<k_i$ and are given in terms of the expressions (23) and (24) by

$$l_{11}(\alpha)=\frac{u_{11}(\alpha)}{i(\kappa+k\beta_1)}+\frac{u_{21}(\alpha)}{i(\kappa+k\beta_2)}, \quad (29)$$

$$l_{12}(\alpha)=\frac{u_{11}(\alpha)(k+\alpha)^{1/2}}{i(\kappa+k\beta_1)}-\frac{u_{21}(\alpha)(k+\alpha)^{1/2}}{i(\kappa+k\beta_2)}, \quad (30)$$

$$l_{21}(\alpha)=\frac{u_{11}(\alpha)\kappa}{\kappa+k\beta_1}-\frac{u_{21}(\alpha)\kappa}{\kappa+k\beta_2}, \quad (31)$$

$$l_{22}(\alpha)=\frac{u_{11}(\alpha)\kappa(k+\alpha)^{1/2}}{\kappa+k\beta_1}+\frac{u_{21}(\alpha)\kappa(k+\alpha)^{1/2}}{\kappa+k\beta_2}. \quad (32)$$

Having factorised $\boldsymbol{K}(\alpha)$ explicitly, we substitute (21) into (17) giving

$$\boldsymbol{U}^{-1}(\alpha)\boldsymbol{\Psi}_{+}(\alpha)=\boldsymbol{L}^{-1}(\alpha)\boldsymbol{\Phi}_{-}(\alpha)+\boldsymbol{U}^{-1}(\alpha)\boldsymbol{D}(\alpha), \tag{33}$$

since $\boldsymbol{U}(\alpha)$ and $\boldsymbol{L}(\alpha)$ are non-singular matrices. By carrying out the matrix multiplication in Eqn. (33) we obtain the two equations

$$(\text{Det } \boldsymbol{U})^{-1}(u_{22}\Psi_1^+ - u_{12}\Psi_2^+) = (\text{Det } \boldsymbol{L})^{-1}(l_{22}\Phi_1^- - l_{12}\Phi_2^-) + G_1, \tag{34}$$

$$(\text{Det } \boldsymbol{U})^{-1}(-u_{21}\Psi_1^+ + u_{11}\Psi_2^+) = (\text{Det } \boldsymbol{L})^{-1}(-l_{21}\Phi_1^- + l_{11}\Phi_2^-) + G_2, \tag{35}$$

where

$$G_1(\alpha) = (\text{Det } \boldsymbol{U})^{-1}\{u_{22}(\alpha)(k\beta_1 - \kappa) + u_{12}(\alpha)(k\beta_2 + \kappa)\} \times \exp[i\{\alpha x_0 + \kappa y_0\}]/(2\kappa), \tag{36}$$

$$G_2(\alpha) = (\text{Det } \boldsymbol{U})^{-1}\{-u_{21}(\alpha)(k\beta_1 - \kappa) - u_{11}(\alpha)(k\beta_2 + \kappa)\} \times \exp[i\{\alpha x_0 + \kappa y_0\}]/(2\kappa), \tag{37}$$

$$\text{Det } \boldsymbol{U}(\alpha) = u_{11}(\alpha)u_{22}(\alpha) - u_{12}(\alpha)u_{21}(\alpha),$$

$$\text{Det } \boldsymbol{L}(\alpha) = l_{11}(\alpha)l_{22}(\alpha) - l_{12}(\alpha)l_{21}(\alpha).$$

By means of Cauchy's integral theorem, see Noble [10], we can let

$$G_1(\alpha) = G_1^+(\alpha) + G_1^-(\alpha), \qquad G_2(\alpha) = G_2^+(\alpha) + G_2^-(\alpha), \tag{38}$$

where

$$G_1^{\pm}(\alpha) = \pm\frac{1}{2\pi i}\int_{-\infty \mp i\tau_1}^{\infty \mp i\tau_1} \frac{G_1(t)}{t-\alpha}\,\mathrm{d}t, \tag{39}$$

$$G_2^{\pm}(\alpha) = \pm\frac{1}{2\pi i}\int_{-\infty \mp i\tau_1}^{\infty \mp i\tau_1} \frac{G_2(t)}{t-\alpha}\,\mathrm{d}t, \qquad 0 < \tau_1 < k_i. \tag{40}$$

The representations (39) and (40) with the upper (lower) sign are valid when $\text{Im}(\alpha) > -\tau_1(\text{Im}(\alpha) < \tau_1)$ and define $G_{1,2}^+(\alpha)(G_{1,2}^-(\alpha))$ as analytic functions in $\text{Im}(\alpha) > -\tau_1(\text{Im}(\alpha) < \tau_1)$. We note that in the limiting case of $\tau_1 = k_i = 0$ the above integrands have an integrable singularity at $t = -k$. This follows from the results (A47) of Appendix A, which show that $G_1(t) = O(1)$, $G_2(t) = O((k+t)^{-1/2})$. Standard asymptotics also shows that

$$G_{1,2}^{\pm}(\alpha) = O(\alpha^{-1}), \qquad \text{as } |\alpha| \to \infty, \tag{41}$$

in their regions of regularity.

We may now write (34) and (35), by means of (38), in the standard Wiener-Hopf forms:

$$(\text{Det } \boldsymbol{U})^{-1}(u_{22}\Psi_1^+ - u_{12}\Psi_2^+) - G_1^+ = (\text{Det } \boldsymbol{L})^{-1}(l_{22}\Phi_1^- - l_{12}\Phi_2^-) + G_1^-, \tag{42}$$

$$(\text{Det } \boldsymbol{U})^{-1}(-u_{21}\Psi_1^+ + u_{11}\Psi_2^+) - G_2^+ = (\text{Det } \boldsymbol{L})^{-1}(-l_{21}\Phi_1^- + l_{11}\Phi_2^-) + G_2^-. \tag{43}$$

In order to be able to apply the normal Wiener-Hopf argument to (42) and (43) we shall require some knowledge of the behaviour of the functions as $|\alpha| \to \infty$.

The edge condition (4) requires that the transformed functions must behave like

$$\begin{aligned} &\Phi_1^-(\alpha) = O(\alpha^{-1}), \quad &&\Phi_2^-(\alpha) = O(\alpha^{-1/2}) \quad \text{for} \quad \operatorname{Im}(\alpha) < k_i, \quad &&|\alpha| \to \infty; \\ &\Psi_1^+(\alpha) = O(\alpha^{-1/2}), \quad &&\Psi_2^+(\alpha) = O(\alpha^{-1/2}) \quad \text{for} \quad \operatorname{Im}(\alpha) > -k_i, \quad &&|\alpha| \to \infty. \end{aligned} \tag{44}$$

By using the above results in conjunction with (41) and the asymptotic growth estimates (A45) and (A46) of Appendix A we find that:
For $\operatorname{Im}(\alpha) > -k_i$ as $|\alpha| \to \infty$,

$$\begin{aligned} &(\operatorname{Det} \boldsymbol{U})^{-1} u_{22} \Psi_1^+ = O(\alpha^{-1/2}), \qquad &&(\operatorname{Det} \boldsymbol{U})^{-1} u_{12} \Psi_2^+ = O(\alpha^{-1/2}), \\ &G_1^+(\alpha) = O(\alpha^{-1}), &&G_2^+(\alpha) = O(\alpha^{-1}), \\ &(\operatorname{Det} \boldsymbol{U})^{-1} u_{21} \Psi_1^+ = O(\alpha^{-1}), &&(\operatorname{Det} \boldsymbol{U})^{-1} u_{11} \Psi_2^+ = O(\alpha^{-1}). \end{aligned} \tag{45}$$

For $\operatorname{Im}(\alpha) < k_i$ as $|\alpha| \to \infty$,

$$\begin{aligned} &(\operatorname{Det} \boldsymbol{L})^{-1} l_{22} \Phi_1^- = O(1), \qquad &&(\operatorname{Det} \boldsymbol{L})^{-1} l_{12} \Phi_2^- = O(\alpha^{-1/2}) \\ &G_1^-(\alpha) = O(\alpha^{-1}), &&G_2^-(\alpha) = O(\alpha^{-1}), \\ &(\operatorname{Det} \boldsymbol{L})^{-1} l_{21} \Phi_1^- = O(\alpha^{-1/2}), &&(\operatorname{Det} \boldsymbol{L})^{-1} l_{11} \Phi_2^- = O(\alpha^{-1}). \end{aligned} \tag{46}$$

The results (45) and (46) show that the left-hand side and right-hand side of the equation (43) are analytic and asymptotic to $o(1)$ as $|\alpha| \to \infty$ in $\operatorname{Im}(\alpha) > -k_i$ and $\operatorname{Im}(\alpha) < k_i$, respectively. Similarly the left-hand side of Eqn. (42) is analytic and asymptotic to $o(1)$ in $\operatorname{Im}(\alpha) > -k_i$ as $|\alpha| \to \infty$, whereas the right-hand side is analytic and asymptotic to $O(1)$ as $|\alpha| \to \infty$ in $\operatorname{Im}(\alpha) < k_i$. Thus by Liouville's theorem the analytic function which is a continuation of both sides of these equations in the entire α-plane is a constant; the constant being zero. Hence

$$\boldsymbol{U}^{-1} \boldsymbol{\Psi}_+ = \boldsymbol{G}_+ = \begin{pmatrix} G_1^+ \\ G_2^+ \end{pmatrix} \Rightarrow \boldsymbol{\Psi}_+ = \boldsymbol{U} \boldsymbol{G}_+$$

or

$$\Psi_1^+ = G_1^+ u_{11} + G_2^+ u_{12}, \tag{47}$$

$$\Psi_2^+ = G_1^+ u_{21} + G_2^+ u_{22}. \tag{48}$$

By substituting (47) and (48) into (15) and (16) we have

$$\begin{aligned} A(\alpha) &= -i(\kappa + k\beta_1)^{-1}(G_1^+ u_{11} + G_2^+ u_{12}) + (\kappa - k\beta_1)(\kappa + k\beta_1)^{-1} \\ &\qquad \times \exp[i\{\alpha x_0 + \kappa y_0\}]/(2i\kappa), \\ B(\alpha) &= i(\kappa + k\beta_2)^{-1}(G_1^+ u_{21} + G_2^+ u_{22}), \end{aligned} \tag{49}$$

which on substituting into (7) and (8) and using the inverse Fourier transform of (5) gives

$$\chi(x, y) = \frac{1}{2\pi} \int_{-\infty + i\tau}^{\infty + i\tau} \left\{ \frac{G_1^+ u_{11} + G_2^+ u_{12}}{i(\kappa + k\beta_1)} + \frac{(\kappa - k\beta_1)\, e^{i(\alpha x_0 + \kappa y_0)}}{(\kappa + k\beta_1) 2i\kappa} \right\} e^{-i\alpha x + i\kappa y} d\alpha$$

$$+ \frac{1}{4i} H_0^{(1)}\left(k\sqrt{(x - x_0)^2 + (y - y_0)^2}\right), \qquad (y > 0), \tag{50}$$

$$= \frac{1}{2\pi} \int_{-\infty + i\tau}^{\infty + i\tau} \left\{ \frac{G_1^+ u_{21} + G_2^+ u_{22}}{-i(\kappa + k\beta_2)} \right\} e^{-i\alpha x - i\kappa y} d\alpha, \qquad (y < 0), \tag{51}$$

$$-k_i < -\tau_1 < \tau < k_i,$$

as the solution to the original boundary value problem. As a check, if we allow $\beta_1 = \beta_2 = 0$ the expressions (50) and (51) reduce to the known result (see Noble [10, p. 87]) for the diffraction of a line source field by a rigid half-plane.

The physical interpretation of the solution given by (50) and (51) is made more apparent by asymptotically evaluating the integrals for the receiver point (x, y) such that $k(x^2 + y^2)^{1/2} \to \infty$. This corresponds to the observer at (x, y) being in the far field. In practice if the line source at (x_0, y_0) and the receiver at (x, y) are more than two wavelengths from the edge (0, 0) of the barrier then to a good approximation we can assume that we are in the far field, and the incident field is a plane wave.

4. Asymptotic expressions for the far field

The asymptotic methods though straightforward are tedious. We shall merely outline the calculations, more details of the techniques can be found in Noble [10]. Consider first $G_{1,2}^+(\alpha)$ as given by (39) and (40); let k be real, then $\tau_1 = 0$ and the integration path along the real axis is indented below the point $t = \alpha$. Substitute $x_0 = r_0 \cos\theta_0$, $y_0 = r_0 \sin\theta_0$, $0 < \theta_0 < \pi$; $t = k\cos\xi$, $0 < \operatorname{Re}\xi < \pi$, then the integrand has a saddle point at $\xi = \theta_0$. The integration path is now deformed into the steepest-descent path $S(\theta_0)$ described by $\operatorname{Re}[\cos(\xi - \theta_0)] = 1$, $\operatorname{Im}[\cos(\xi - \theta_0)] \geqslant 0$. In the deformation the pole at $k\cos\xi = \alpha$ is intercepted if $\alpha < k\cos\theta_0$. The integral along $S(\theta_0)$ is asymptotically expanded as $kr_0 \to \infty$ by means of the saddle point method.

Thus it is found that

$$G_1^+(\alpha) \sim \frac{1}{4\pi i} \frac{(\beta_1 - \sin\theta_0) u_{22}(k\cos\theta_0) + u_{12}(k\cos\theta_0)(\beta_2 + \sin\theta_0)}{\operatorname{Det} U(k\cos\theta_0)(\cos\theta_0 - \alpha/k)}$$

$$\times \sqrt{\frac{2\pi}{kr_0}}\, e^{ikr_0 - i\pi/4}$$

$$+ \frac{(k\beta_1 - \kappa) u_{22}(\alpha) + (k\beta_2 + \kappa) u_{12}(\alpha)}{2\kappa \operatorname{Det} U(\alpha)} H[k\cos\theta_0 - \alpha]\, e^{i(\alpha x_0 + \kappa y_0)}, \tag{52}$$

$$G_2^+(\alpha) \sim \frac{-1}{4\pi i} \frac{(\beta_1 - \sin\theta_0)u_{21}(k\cos\theta_0) + (\beta_2 + \sin\theta_0)u_{11}(k\cos\theta_0)}{\text{Det}\, U(k\cos\theta_0)(\cos\theta_0 - \alpha/k)}$$

$$\times \sqrt{\frac{2\pi}{kr_0}}\, \mathrm{e}^{ikr_0 - i\pi/4}$$

$$- \frac{(k\beta_1 - \kappa)u_{21}(\alpha) + (k\beta_2 + \kappa)u_{11}(\alpha)}{2\kappa\, \text{Det}\, U(\alpha)} H[k\cos\theta_0 - \alpha]\, \mathrm{e}^{i(\alpha x_0 + \kappa y_0)}, \quad (53)$$

where $H[x] = 1$ for $x > 0$, $H[x] = 0$ for $x < 0$ (Heaviside step function); the result is valid for $kr_0 \to \infty$, $-k < \alpha < k$; the second terms arise from the residue contributions.

The results (52) and (53) for $G_{1,2}^+(\alpha)$, when inserted into (50) and (51), give

$$\chi(x, y) = \chi_D(x, y) + \chi_{GA}(x, y), \quad (54)$$

where

$$\chi_D(x, y) = \sqrt{\frac{2\pi}{kr_0}} \frac{\mathrm{e}^{ikr_0 - i\pi/4}}{4\pi i} \frac{1}{2\pi}$$

$$\times \int_{-\infty + i\tau}^{\infty + i\tau} [(\beta_1 - \sin\theta_0)u_{22}(k\cos\theta_0)u_{11}(\alpha) + (\beta_2 + \sin\theta_0)u_{12}(k\cos\theta_0)u_{11}(\alpha)$$

$$-(\beta_1 - \sin\theta_0)u_{21}(k\cos\theta_0)u_{12}(\alpha) - (\beta_2 + \sin\theta_0)u_{11}(k\cos\theta_0)u_{12}(\alpha)]$$

$$\times [\text{Det}\, U(k\cos\theta_0)(\cos\theta_0 - \alpha/k)i(\kappa + k\beta_1)]^{-1}\, \mathrm{e}^{-i\alpha x + i\kappa y} \mathrm{d}\alpha, \qquad (y > 0), \quad (55)$$

$$= \sqrt{\frac{2\pi}{kr_0}} \frac{\mathrm{e}^{ikr_0 - i\pi/4}}{4\pi i} \frac{1}{2\pi}$$

$$\times \int_{\infty + i\tau}^{\infty + i\tau} [(\beta_1 - \sin\theta_0)u_{22}(k\cos\theta_0)u_{21}(\alpha) + (\beta_2 + \sin\theta_0)u_{12}(k\cos\theta_0)u_{21}(\alpha)$$

$$-(\beta_1 - \sin\theta_0)u_{21}(k\cos\theta_0)u_{22}(\alpha) - (\beta_2 + \sin\theta_0)u_{11}(k\cos\theta_0)u_{22}(\alpha)]$$

$$\times [\text{Det}\, U(k\cos\theta_0)(\cos\theta_0 - \alpha/k)(-i)(\kappa + k\beta_2)]^{-1}\, \mathrm{e}^{-i\alpha x - i\kappa y} \mathrm{d}\alpha, \qquad (y < 0); \quad (56)$$

$$\chi_{GA}(x, y) = \frac{1}{4i} H_0^{(1)}\left(k\sqrt{(x - x_0)^2 + (y - y_0)^2}\right)$$

$$+ \frac{1}{2\pi} \int_{-\infty + i\tau}^{\infty + i\tau} \left(\frac{k\beta_1 - \kappa}{k\beta_1 + \kappa}\right) \{H[k\cos\theta_0 - \alpha] - 1\} \frac{\mathrm{e}^{-i\alpha(x - x_0) + i\kappa(y + y_0)}}{2i\kappa} \mathrm{d}\alpha,$$

$$(y > 0), \quad (57)$$

$$= \frac{1}{2\pi} \int_{-\infty + i\tau}^{\infty + i\tau} H[k\cos\theta_0 - \alpha] \frac{\mathrm{e}^{-i\alpha(x - x_0) + i\kappa(y_0 - y)}}{2i\kappa} \mathrm{d}\alpha, \qquad (y < 0). \quad (58)$$

The integrals in the expressions (55) and (56) can be asymptotically expanded for $kr \to \infty$ by the saddle point method following the usual steps: substitute $x = r\cos\theta$, $y = r\sin\theta$, $-\pi < \theta < \pi$; $\alpha = k\cos\xi$, $0 < \operatorname{Re}\xi < \pi$, then the integrand has a saddle point at $\xi = \pi - \theta$ and $\xi = \pi + \theta$, respectively; deform the path of integration into $S(\pi - \theta)$ and $S(\pi + \theta)$, respectively; apply the saddle point formula. This gives

$$\chi_D(r\cos\theta, r\sin\theta) \sim \frac{1}{4i}\sqrt{\frac{2}{\pi k r_0}}\, e^{ikr_0 - i\pi/4} D(\theta, \theta_0)\frac{e^{ikr}}{\sqrt{r}} \tag{59}$$

where

$$D(\theta, \theta_0) = -\frac{e^{i\pi/4}}{2\sqrt{2\pi k}} \frac{\exp\left[\frac{1}{2}\int_{k\cos\theta_0}^{-k\cos\theta} Q(u)\mathrm{d}u\right]}{(\cos\theta + \cos\theta_0)\cos(\theta_0/2)}$$

$$\times \frac{|\sin\theta|}{\{(\sin\theta + \beta_1)(\sin\theta - \beta_2)\}^{1/2}}\left(\frac{|\sin\theta| - \beta_2}{|\sin\theta| + \beta_1}\right)^{1/2}$$

$$\left[\frac{\left(\sqrt{2}\sin\theta/2 + \sqrt{B_1(+)}\right)\left(\sqrt{2}\sin\theta/2 + \sqrt{B_1(-)}\right)}{\left(\sqrt{2}\sin\theta/2 + \sqrt{B_2(+)}\right)\left(\sqrt{2}\sin\theta/2 + \sqrt{B_2(-)}\right)}\right]^{1/2}$$

$$\times\left\{(\beta_1 - \sin\theta_0)\left(\sin\frac{\theta}{2} + \cos\frac{\theta_0}{2}\right)\right.$$

$$\left[\frac{\left(\sqrt{2}\cos\theta_0/2 + \sqrt{B_2(+)}\right)\left(\sqrt{2}\cos\theta_0/2 + \sqrt{B_2(-)}\right)}{\left(\sqrt{2}\cos\theta_0/2 + \sqrt{B_1(+)}\right)\left(\sqrt{2}\cos\theta_0/2 + \sqrt{B_1(-)}\right)}\right]^{1/2}$$

$$+ (\beta_2 + \sin\theta_0)\left(\cos\frac{\theta_0}{2} - \sin\frac{\theta}{2}\right)$$

$$\left.\left[\frac{\left(\sqrt{2}\cos\theta_0/2 + \sqrt{B_1(+)}\right)\left(\sqrt{2}\cos\theta_0/2 + \sqrt{B_1(-)}\right)}{\left(\sqrt{2}\cos\theta_0/2 + \sqrt{B_2(+)}\right)\left(\sqrt{2}\cos\theta_0/2 + \sqrt{B_2(-)}\right)}\right]^{1/2}\right\}, \tag{60}$$

$$kr \to \infty, \quad kr_0 \to \infty, \quad 0 < \theta_0 < \pi, \quad -\pi < \theta < \pi, \quad \cos\theta + \cos\theta_0 \neq 0.$$

In a similar fashion the integrals appearing in the expressions (57) and (58) can be asymptotically evaluated by the saddle point method. In the integrand of the expression (57) let $x - x_0 = R_2\cos\theta_2$, $y + y_0 = R_2\sin\theta_2$, $0 < \theta_2 < \pi$, $\alpha = k\cos\xi$, $0 < \operatorname{Re}\xi < \pi$; and in the expression (58) let $x - x_0 = R_1\cos\theta_1$, $y - y_0 = -R_1\sin\theta_1$, $0 < \theta_1 < \pi$, $\alpha = k\cos\xi$, $0 < \operatorname{Re}\xi < \pi$ (see Fig. 1). The saddle point of (57) and (58) is then given by $\xi = \pi - \theta_2$ and

$\xi = \pi - \theta_1$, respectively. Deforming the path of integration into $S(\pi - \theta_2)$ and $S(\pi - \theta_1)$, respectively, and applying the saddle point formula gives

$$\chi_{GA}(r\cos\theta, r\sin\theta) \sim \frac{1}{4i}\sqrt{\frac{2}{\pi k R_1}}\, e^{ikR_1 - i\pi/4} + \frac{1}{4i}\sqrt{\frac{2}{\pi k R_2}}\, e^{ikR_2 - i\pi/4}\left(\frac{\beta_1 - \sin\theta_2}{\beta_1 + \sin\theta_2}\right)$$

$$\times\{H[\cos\theta_0 + \cos\theta_2] - 1\}, \qquad 0 < \theta < \pi,\ kR_2 \to \infty;$$

$$\sim \frac{1}{4i}\sqrt{\frac{2}{\pi k R_1}}\, e^{ikR_1 - i\pi/4} H[\cos\theta_0 + \cos\theta_1], \qquad -\pi < \theta < 0, \qquad kR_1 \to \infty.$$

By using the fact that

$$H[\cos\theta_0 + \cos\theta_2] - 1 = -H[\theta + \theta_0 - \pi],$$

$$H[\cos\theta_0 + \cos\theta_1] = H[\theta - \theta_0 + \pi]H[-\theta]$$

for $0 < \theta_{1,2} < \pi$, $0 < \theta_0 < \pi$, we can rewrite the above expression for χ_{GA} as

$$\chi_{GA}(r\cos\theta, r\sin\theta) \sim \frac{1}{4i}\sqrt{\frac{2}{\pi k R_1}}\, e^{ikR_1 - i\pi/4} H[\theta - \theta_0 + \pi]$$

$$-\frac{1}{4i}\sqrt{\frac{2}{\pi k R_2}}\, e^{ikR_2 - i\pi/4}\left(\frac{\beta_1 - \sin\theta_2}{\beta_1 + \sin\theta_2}\right) H[\theta + \theta_0 - \pi], \tag{61}$$

$$-\pi < \theta < \pi, \qquad kR_1 \to \infty, \qquad kR_2 \to \infty.$$

If the expressions (59) and (61) are substituted into (54) we have finally the expression for the far field

$$\chi(r\cos\theta, r\sin\theta) = \frac{1}{4i} H_0^{(1)}(kR_1) H[\theta - \theta_0 + \pi]$$

$$+ \frac{1}{4i} H_0^{(1)}(kR_2)\left(\frac{\sin\theta_2 - \beta_1}{\sin\theta_2 + \beta_1}\right) H[\theta + \theta_0 - \pi]$$

$$+ \frac{1}{4i} H_0^{(1)}(kr_0) D(\theta, \theta_0) \frac{e^{ikr}}{\sqrt{r}}, \tag{62}$$

$$kr \to \infty, \qquad kr_0 \to \infty, \qquad -\pi < \theta < \pi, \qquad 0 < \theta_0 < \pi, \qquad \theta \neq \pm(\pi - \theta_0),$$

where $D(\theta, \theta_0)$ is given by (60) and we have replaced the asymptotic term $(2/\pi z)^{1/2} \exp[iz - i\pi/4]$ by the Hankel function $H_0^{(1)}(z)$; it is understood that its asymptotic form is used in (62).

The physical interpretation of the result (62) in conjunction with Fig. 1 is now obvious. The first term represents the incident cylindrical wave due to a line source at (x_0, y_0). The second term is the wave reflected from the upper impedance face of the half-plane. This reflected wave appears to radiate from an image line source at $(x_0, -y_0)$. The reflection coefficient $(\sin\theta_2 - \beta_1)/(\sin\theta_2 + \beta_1)$ occurring in the second term, is the same as for reflection of a plane wave incident at an angle $\pi - \theta_2$ on an infinite absorbent plane. The first two terms of the expression (62) represent the geometrical acoustic field and they will not exist everywhere. The regions where they are present are governed by the Heaviside step functions which multiply the Hankel functions. Physically these regions correspond to the shadow and insonified regions. On the boundary between these regions the arguments of the Heaviside step functions vanish. The last term of the expression (62) represents the diffracted field, which is a cylindrical wave which appears to radiate from the edge of the half plane, to all points in space.

Acknowledgement

I would like to thank Professor J. Boersma of Eindhoven University of Technology for his considerable help in the preparation of this paper.

Appendix A

In this appendix we shall factorise the matrix $\boldsymbol{K}(\alpha)$ given by the expression (19) of the main text. In order to simplify the formulae of this appendix we shall assume k is real, i.e. $k_i = 0$. There is no loss of generality in this assumption. The end results are analytic functions of k which will be valid for $k_i \geqslant 0$ by analytic continuation. We shall reduce the problem of factorisation to the solution of a set of Hilbert problems. These Hilbert problems are then solved by Muskhelishvili's theory [11]. Some asymptotic growth estimates conclude this appendix.

Reduction of the matrix factorisation problem to Hilbert problems

We assume a factorisation of the form

$$\boldsymbol{K}(\alpha) = \boldsymbol{U}(\alpha)\boldsymbol{L}^{-1}(\alpha), \tag{A1}$$

exists where

$$\boldsymbol{L}(\alpha) = \begin{pmatrix} l_{11}(\alpha) & l_{12}(\alpha) \\ l_{21}(\alpha) & l_{22}(\alpha) \end{pmatrix}, \tag{A2}$$

$$\boldsymbol{U}(\alpha) = \begin{pmatrix} u_{11}(\alpha) & u_{12}(\alpha) \\ u_{21}(\alpha) & u_{22}(\alpha) \end{pmatrix}. \tag{A3}$$

The elements l_{ij}, $(i, j = 1, 2)$ of $\boldsymbol{L}(\alpha)$ are assumed to be analytic in the cut α-plane $|\arg(k-\alpha)| < \pi$. The elements $u_{ij}(i, j = 1, 2)$ of $\boldsymbol{U}(\alpha)$ are assumed to be analytic in the cut α-plane $|\arg(k+\alpha)| < \pi$. This means that $\boldsymbol{L}(\alpha)$ is analytic everywhere except along the branch cut $k \leqslant \alpha < \infty$, $\mathrm{Im}(\alpha) = 0$; and $U(\alpha)$ is analytic everywhere except along the branch cut $-\infty < \alpha \leqslant -k$, $\mathrm{Im}(\alpha) = 0$.

We note from the expression (19), that

$$\mathrm{Det}\, \boldsymbol{K}(\alpha) = -i(\kappa + k\beta_1)(\kappa + k\beta_2)/(2\kappa) \neq 0, \tag{A4}$$

in the cut α-plane, since $-\pi/2 < \arg(\kappa) < \pi/2$ and $\mathrm{Re}(k\beta_1) \geqslant 0$, $\mathrm{Re}(k\beta_2) \geqslant 0$. Hence $\boldsymbol{K}(\alpha)$, and consequently $\boldsymbol{U}(\alpha)$ and $\boldsymbol{L}^{-1}(\alpha)$ are non-singular matrices in the cut plane. We now analytically evaluate the left hand side, and consequently the right-hand side of (A1) about the branch cut at $\alpha = -k$. This gives

$$\left.\begin{aligned} \boldsymbol{K}^+(\xi) &= \boldsymbol{U}^+(\xi)\boldsymbol{L}^{-1}(\xi) \\ \boldsymbol{K}^-(\xi) &= \boldsymbol{U}^-(\xi)\boldsymbol{L}^{-1}(\xi) \end{aligned}\right\} -\infty < \xi < -k \tag{A5}$$

where in this appendix only we use the notation $F^+(\xi) = F(|\xi|\,\mathrm{e}^{i\pi})$ to denote values of F on the upper side of the cut, and $F^-(\xi) = F(|\xi|\,\mathrm{e}^{-i\pi})$ to denote values of F on the lower side of the cut. We remark that in (A5) $\boldsymbol{L}^{-1}(\alpha)$ does not jump in value on crossing this cut because it is analytic at $\alpha = \xi$, $-\infty < \xi \leqslant -k$. Eliminating $\boldsymbol{L}^{-1}(\xi)$ in the expression (A5) gives

$$\boldsymbol{U}^+(\xi) = \boldsymbol{K}^+(\xi)\left[\boldsymbol{K}^{-1}(\xi)\right]^- \boldsymbol{U}^-(\xi), \qquad -\infty < \xi < -k. \tag{A6}$$

We note that on the cut α-plane

$$\kappa = \pm i\left(\xi^2 - k^2\right)^{1/2} = \pm i|\kappa| \quad \text{for} \quad \alpha = -\xi\,\mathrm{e}^{\pm i\pi}, \qquad -\infty < \xi < -k, \tag{A7}$$

and therefore

$$\boldsymbol{K}^+(\xi) = \tfrac{1}{2}\begin{pmatrix} (-|\kappa| + ik\beta_1) & -(-|\kappa| + ik\beta_1)/|\kappa| \\ (-|\kappa| + ik\beta_2) & (-|\kappa| + ik\beta_2)/|\kappa| \end{pmatrix}, \tag{A8}$$

$$\left[\boldsymbol{K}^{-1}(\xi)\right]^- = \begin{pmatrix} (|\kappa| + ik\beta_1)^{-1} & (|\kappa| + ik\beta_2)^{-1} \\ |\kappa|(|\kappa| + ik\beta_1)^{-1} & -|\kappa|(|\kappa| + ik\beta_2)^{-1} \end{pmatrix}, \tag{A9}$$

so that

$$\boldsymbol{K}^+(\xi)\left[\boldsymbol{K}^{-1}(\xi)\right]^- = \begin{pmatrix} 0 & \dfrac{-|\kappa| + ik\beta_1}{|\kappa| + ik\beta_2} \\ \dfrac{-|\kappa| + ik\beta_2}{|\kappa| + ik\beta_1} & 0 \end{pmatrix}. \tag{A10}$$

Substituting (A10) into (A6) gives

$$u_{11}^{+}(\xi)=\left(\frac{-|\kappa|+ik\beta_1}{|\kappa|+ik\beta_2}\right)u_{21}^{-}(\xi), \tag{A11}$$

$$u_{21}^{+}(\xi)=\left(\frac{-|\kappa|+ik\beta_2}{|\kappa|+ik\beta_1}\right)u_{11}^{-}(\xi), \tag{A12}$$

$$u_{12}^{+}(\xi)=\left(\frac{-|\kappa|+ik\beta_1}{|\kappa|+ik\beta_2}\right)u_{22}^{-}(\xi), \tag{A13}$$

$$u_{22}^{+}(\xi)=\left(\frac{-|\kappa|+ik\beta_2}{|\kappa|+ik\beta_1}\right)u_{12}^{-}(\xi), \qquad -\infty<\xi<-k. \tag{A14}$$

Equations (A11) and (A12) form a coupled system of Hilbert problems for u_{11} and u_{21}. Similarly Eqns. (A13) and (A14) form a coupled system of Hilbert problems for u_{12} and u_{22}.

Clearly if we can solve the coupled Hilbert problems

$$u_1^{+}(\xi)=\left(\frac{ik\beta_1-|\kappa|}{ik\beta_2+|\kappa|}\right)u_2^{-}(\xi), \tag{A15}$$

$$-\infty<\xi<-k,$$

$$u_2^{+}(\xi)=\left(\frac{ik\beta_2-|\kappa|}{ik\beta_1+|\kappa|}\right)u_1^{-}(\xi), \tag{A16}$$

then we can solve Eqns. (A11) to (A14).

Solution of the Hilbert problems (A15) and (A16)

We solve (A15) and (A16) by first uncoupling them. This is achieved by taking logarithms of (A15) and (A16) and then adding and subtracting the resulting equations. This gives the two uncoupled equations

$$[\log V(\xi)]^{+}-[\log V(\xi)]^{-}=\log\left[\left(\frac{ik\beta_1-|\kappa|}{ik\beta_1+|\kappa|}\right)\left(\frac{ik\beta_2-|\kappa|}{ik\beta_2+|\kappa|}\right)\right], \tag{A17}$$

$$[\log W(\xi)]^{+}+[\log W(\xi)]^{-}=\log\left[\frac{|\kappa|^2+k^2\beta_1^2}{|\kappa|^2+k^2\beta_2^2}\right], \qquad -\infty<\xi<-k; \tag{A18}$$

where

$$V(\xi)=u_1(\xi)u_2(\xi), \tag{A19}$$

$$W(\xi)=u_1(\xi)/u_2(\xi). \tag{A20}$$

By using the result $[\sqrt{k+\alpha}\,]^{\pm} = \pm i|k+\xi|^{1/2}$ in the equation (A18), we can write (A17) and (A18) in the form

$$[\log V(\xi)]^{+} - [\log V(\xi)]^{-} = \log\left[\left(\frac{|\kappa| - ik\beta_1}{|\kappa| + ik\beta_1}\right)\left(\frac{|\kappa| - ik\beta_2}{|\kappa| + ik\beta_2}\right)\right], \tag{A21}$$

$-\infty < \xi < -k,$

$$\left[\frac{\log W(\xi)}{\sqrt{k+\xi}}\right]^{+} - \left[\frac{\log W(\xi)}{\sqrt{k+\xi}}\right]^{-} = \frac{-i}{|k+\xi|^{1/2}}\log\left[\frac{|\kappa|^2 + k^2\beta_1^2}{|\kappa|^2 + k^2\beta_2^2}\right]. \tag{A22}$$

These are standard Hilbert problems whose solution is given (see [11]) by

$$V(\alpha) = \exp\left[\frac{1}{2\pi i}\int_{-\infty}^{-k}\log\left[\left(\frac{|\kappa| - ik\beta_1}{|\kappa| + ik\beta_1}\right)\left(\frac{|\kappa| - ik\beta_2}{|\kappa| + ik\beta_2}\right)\right]\frac{\mathrm{d}t}{t-\alpha}\right], \tag{A23}$$

$$W(\alpha) = \exp\left[-\frac{(k+\alpha)^{1/2}}{2\pi}\int_{-\infty}^{-k}\frac{1}{|k+t|^{1/2}}\log\left[\frac{|\kappa|^2 + k^2\beta_1^2}{|\kappa|^2 + k^2\beta_2^2}\right]\frac{\mathrm{d}t}{t-\alpha}\right], \tag{A24}$$

Obviously the exponents of $V(\alpha)$ and $W(\alpha)$, and consequently $V(\alpha)$ and $W(\alpha)$, are analytic in $|\arg(k+\alpha)| < \pi$; furthermore $V(\alpha) \neq 0$ and $W(\alpha) \neq 0$ in $|\arg(k+\alpha)| < \pi$. The expressions (A23) and (A24) can be reduced to simpler form by carrying out the integrations, see Appendix B. In particular it is shown there that

$$W(\alpha) = O(1) \quad \text{and} \quad V(\alpha) = O(1) \quad \text{as} \quad |\alpha| \to \infty, \qquad |\arg(k+\alpha)| < \pi; \tag{A25}$$

and

$$W(\alpha) = O(1) \quad \text{and} \quad V(\alpha) = O\left((k+\alpha)^{-1}\right) \quad \text{as} \quad \alpha \to -k, \qquad \mathrm{Re}(\beta_{1,2}) > 0. \tag{A26}$$

Thus particular solutions of (A15) and (A16) are given, from (A19) and (A20)

$$u_1(\alpha) = -[V(\alpha)]^{1/2}[W(\alpha)]^{1/2}, \tag{A27}$$

$$u_2(\alpha) = [V(\alpha)]^{1/2}[W(\alpha)]^{-1/2}, \tag{A28}$$

where

$$[V(\alpha)]^{1/2} = \exp[\tfrac{1}{2}J(\alpha)], \tag{A29}$$

$$[W(\alpha)]^{1/2} = \left[\frac{\left(\sqrt{k+\alpha} + \sqrt{kB_1(+)}\right)\left(\sqrt{k+\alpha} + \sqrt{kB_1(-)}\right)}{\left(\sqrt{k+\alpha} + \sqrt{kB_2(+)}\right)\left(\sqrt{k+\alpha} + \sqrt{kB_2(-)}\right)}\right]^{1/2}; \tag{A30}$$

$J(\alpha)$ is given by the expression (B13) of Appendix B. The choice of sign, on taking the square roots, for $u_1(\alpha)$ and $u_2(\alpha)$ in (A27) to (A30) is justified as follows. With the signs given by (A27) to (A30) we have

$$\frac{u_1^+(\xi)}{u_2^-(\xi)} = -\left(\frac{V^+(\xi)}{V^-(\xi)}\right)^{1/2} [W^+(\xi)W^-(\xi)]^{1/2}.$$

By means of Plemelj's formula [11, §17], (A23) gives

$$\frac{V^+(\xi)}{V^-(\xi)} = \left(\frac{|\kappa| - ik\beta_1}{|\kappa| + ik\beta_1}\right)\left(\frac{|\kappa| - ik\beta_2}{|\kappa| + ik\beta_2}\right), \qquad -\infty < \xi < -k,$$

and from (A30)

$$[W^+(\xi)W^-(\xi)]^{1/2} = \left[\frac{(|k+\xi| + kB_1(+))(|k+\xi| + kB_1(-))}{(|k+\xi| + kB_2(+))(|k+\xi| + kB_2(-))}\right]^{1/2}$$

$$= \left[\frac{\xi^2 - k^2 + k^2\beta_1^2}{\xi^2 - k^2 + k^2\beta_2^2}\right]^{1/2} = \left(\frac{(|\kappa| + ik\beta_1)(|\kappa| - ik\beta_1)}{(|\kappa| + ik\beta_2)(|\kappa| - ik\beta_2)}\right)^{1/2},$$

$$-\infty < \xi < -k.$$

Hence

$$-\left(\frac{V^+(\xi)}{V^-(\xi)}\right)^{1/2} [W^+(\xi)W^-(\xi)]^{1/2} = \frac{ik\beta_1 - |\kappa|}{ik\beta_2 + |\kappa|},$$

and therefore

$$\frac{u_1^+(\xi)}{u_2^-(\xi)} = \frac{ik\beta_1 - |\kappa|}{ik\beta_2 + |\kappa|},$$

which is clearly consistent with (A15).

It is emphasized that the above result for $u_1(\alpha)$ and $u_2(\alpha)$ is just a particular solution, and not the general solution. To obtain the general solution we must impose further conditions on the functions $u_1(\alpha)$ and $u_2(\alpha)$ that we are interested in. First it is required that

$$u_1(\alpha) = O\big((k+\alpha)^{\delta_1}\big), \qquad u_2(\alpha) = O\big((k+\alpha)^{1/2+\delta_2}\big), \qquad \text{as } \alpha \to -k, \qquad \text{(A32)}$$

for some $\delta_{1,2} > -1$, in order to guarantee the convergence of the integrals (39) and (40), the singularity at $t = -k$ being integrable. Secondly, it is customary for the Hilbert problem to require that $u_1(\alpha)$ and $u_2(\alpha)$ have finite degree at infinity, that is, $u_1(\alpha)$ and $u_2(\alpha)$ have polynomial growth as $|\alpha| \to \infty$.

To determine the general solution for $u_1(\alpha)$ and $u_2(\alpha)$ under these conditions, substitute

$$u_1(\alpha) = -[V(\alpha)]^{1/2}[W(\alpha)]^{1/2}u_1^*(\alpha), \qquad u_2(\alpha) = [V(\alpha)]^{1/2}[W(\alpha)]^{-1/2}u_2^*(\alpha), \tag{A33}$$

into the equations (A15) and (A16), leading to the vector Hilbert problem

$$[u_1^*(\xi)]^+ = [u_2^*(\xi)]^-, \qquad [u_2^*(\xi)]^+ = [u_1^*(\xi)]^-, \qquad -\infty < \xi < -k$$

under the conditions

$$u_1^*(\alpha) = O\big((k+\alpha)^{\delta_1+1/2}\big), \qquad u_2^*(\alpha) = O\big((k+\alpha)^{\delta_2+1}\big), \qquad \text{as } \alpha \to -k,$$

and $u_1^*(\alpha)$, $u_2^*(\alpha)$ have finite degree at infinity.

The Hilbert problem is uncoupled through addition and subtraction, viz.

$$[u_1^*(\xi) + u_2^*(\xi)]^+ = [u_1^*(\xi) + u_2^*(\xi)]^-, \qquad -\infty < \xi < -k,$$

$$\left[\frac{u_1^*(\xi) - u_2^*(\xi)}{\sqrt{k+\xi}}\right]^+ = \left[\frac{u_1^*(\xi) - u_2^*(\xi)}{\sqrt{k+\xi}}\right]^-, \qquad -\infty < \xi < -k.$$

where the second equation was divided by $[\sqrt{k+\xi}\,]^{\pm} = \pm i|k+\xi|^{1/2}$. Thus the functions $u_1^*(\alpha) + u_2^*(\alpha)$ and $[u_1^*(\alpha) - u_2^*(\alpha)]/\sqrt{k+\alpha}$ are continuous across the branch cut, hence by a well-known theorem [11, p. 36] these functions are analytic in the entire α-plane except possibly at $\alpha = -k$. Such a possibility is ruled out by the requirement $\delta_{1,2} > -1$, which ensures that there can be no pole singularity at $\alpha = -k$. In conclusion, $u_1^*(\alpha) + u_2^*(\alpha)$ and $[u_1^*(\alpha) - u_2^*(\alpha)]/\sqrt{k+\alpha}$ are entire functions. The second requirement of $u_1^*(\alpha)$, $u_2^*(\alpha)$ having finite degree at infinity, combined with Liouville's theorem then yields

$$u_1^*(\alpha) + u_2^*(\alpha) = 2P_1(\alpha), \qquad \frac{u_1^*(\alpha) - u_2^*(\alpha)}{\sqrt{k+a}} = 2P_2(\alpha),$$

$$u_1^*(\alpha) = P_1(\alpha) + P_2(\alpha)\sqrt{k+\alpha}\,, \qquad u_2^*(\alpha) = P_1(\alpha) - P_2(\alpha)\sqrt{k+\alpha}\,, \tag{A34}$$

where $P_1(\alpha)$, $P_2(\alpha)$ are arbitrary polynomials. Finally the general solution for $u_1(\alpha)$ and $u_2(\alpha)$ is given by (A33) and (A34) as

$$u_1(\alpha) = -[V(\alpha)]^{1/2}[W(\alpha)]^{1/2}\{P_1(\alpha) + P_2(\alpha)\sqrt{k+\alpha}\,\}, \tag{A35}$$

$$u_2(\alpha) = [V(\alpha)]^{1/2}[W(\alpha)]^{-1/2}\{P_1(\alpha) - P_2(\alpha)\sqrt{k+\alpha}\,\}. \tag{A36}$$

Solution of the equations (A11) to (A14)

From the solutions (A35) and (A36) the matrix elements $u_{ij}(\alpha)$, which satisfy Eqns. (A11) to (A14), are given by

$$u_{11}(\alpha) = -[V(\alpha)]^{1/2}[W(\alpha)]^{1/2}\{P_{11}(\alpha) + P_{21}(\alpha)\sqrt{k+\alpha}\},$$

$$u_{21}(\alpha) = [V(\alpha)]^{1/2}[W(\alpha)]^{-1/2}\{P_{11}(\alpha) - P_{21}(\alpha)\sqrt{k+\alpha}\},$$

$$u_{12}(\alpha) = -[V(\alpha)]^{1/2}[W(\alpha)]^{1/2}\{P_{12}(\alpha) + P_{22}(\alpha)\sqrt{k+\alpha}\},$$

$$u_{22}(\alpha) = [V(\alpha)]^{1/2}[W(\alpha)]^{-1/2}\{P_{12}(\alpha) - P_{22}(\alpha)\sqrt{k+\alpha}\},$$

where $P_{ij}(\alpha)$ $(i, j = 1, 2)$ are as yet arbitrary polynomials. The matrix $\boldsymbol{U}(\alpha)$ can be written more compactly as

$$\boldsymbol{U}(\alpha) = \boldsymbol{U}^{(0)}(\alpha)\boldsymbol{P}(\alpha),$$

where

$$\boldsymbol{U}^{(0)}(\alpha) = \begin{pmatrix} -[V(\alpha)]^{1/2}[W(\alpha)]^{1/2} & -[V(\alpha)]^{1/2}[W(\alpha)]^{1/2}(k+\alpha)^{1/2} \\ [V(\alpha)]^{1/2}[W(\alpha)]^{-1/2} & -[V(\alpha)]^{1/2}[W(\alpha)]^{-1/2}(k+\alpha)^{1/2} \end{pmatrix}.$$

Finally we must ensure that $\boldsymbol{U}(\alpha)$ and $\boldsymbol{L}(\alpha)$ are non-singular in the cut α-plane. This puts some restrictions on the P_{ij}. The exact restrictions are determined by looking at Det $\boldsymbol{U}(\alpha)$ and Det $\boldsymbol{L}(\alpha)$. Thus

$$\text{Det } \boldsymbol{U} = \text{Det } \boldsymbol{U}^{(0)} \text{ Det } \boldsymbol{P} = 2V(\alpha)(k+\alpha)^{1/2} \text{ Det } \boldsymbol{P},$$

$$\text{Det } \boldsymbol{L} = \text{Det } \boldsymbol{K}^{-1} \text{ Det } \boldsymbol{U} = \frac{2i\kappa}{(\kappa + k\beta_1)(\kappa + k\beta_2)} 2V(\alpha)(k+\alpha)^{1/2} \text{ Det } \boldsymbol{P}.$$

Therefore $\boldsymbol{U}$ and $\boldsymbol{L}$ will be non singular in the cut α-plane if Det $\boldsymbol{P} \neq 0$ for all α. Since Det $\boldsymbol{P}$ is a polynomial, one must have Det $\boldsymbol{P}$ = constant, i.e. a polynomial of zero degree. The matrix factorisation is not unique, and it is desirable that the polynomials $P_{ij}(\alpha)$ have lowest possible degree, in order that the two sides of the split equations (42) and (43) also have lowest possible degree at infinity. Then the best choice for $\boldsymbol{P}(\alpha)$ is

$$\boldsymbol{P}(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus we have chosen the factorisation

$$u_{11}(\alpha) = -[V(\alpha)]^{1/2}[W(\alpha)]^{1/2}, \quad \text{(A37)}$$

$$u_{21}(\alpha) = [V(\alpha)]^{1/2}[W(\alpha)]^{-1/2}, \quad \text{(A38)}$$

$$u_{12}(\alpha) = -[V(\alpha)]^{1/2}[W(\alpha)]^{1/2}(k+\alpha)^{1/2}, \quad \text{(A39)}$$

$$u_{22}(\alpha) = -[V(\alpha)]^{1/2}[W(\alpha)]^{-1/2}(k+\alpha)^{1/2}, \quad \text{(A40)}$$

where $[V(\alpha)]^{1/2}$ and $[W(\alpha)]^{1/2}$ are given by (A29) and (A30). The corresponding matrix $\boldsymbol{L}(\alpha)$ is obtained by substituting the above expressions (A37) to (A40) into

$$\boldsymbol{L}(\alpha)=\boldsymbol{K}^{-1}(\alpha)\boldsymbol{U}(\alpha)$$

or

$$\begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix}=\begin{pmatrix} -i(\kappa+k\beta_1)^{-1} & -i(\kappa+k\beta_2)^{-1} \\ \kappa(\kappa+k\beta_1)^{-1} & -\kappa(\kappa+k\beta_2)^{-1} \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix},$$

giving

$$l_{11}=\frac{i[V(\alpha)]^{1/2}[W(\alpha)]^{1/2}}{\kappa+k\beta_1}-\frac{i[V(\alpha)]^{1/2}[W(\alpha)]^{-1/2}}{\kappa+k\beta_2}, \tag{A41}$$

$$l_{12}=\frac{i[V(\alpha)]^{1/2}[W(\alpha)]^{1/2}(k+\alpha)^{1/2}}{\kappa+k\beta_1}+\frac{i[V(\alpha)]^{1/2}[W(\alpha)]^{-1/2}(k+\alpha)^{1/2}}{\kappa=k\beta_2}, \tag{A42}$$

$$l_{21}=\frac{-\kappa[V(\alpha)]^{1/2}[W(\alpha)]^{1/2}}{\kappa+k\beta_1}-\frac{\kappa[V(\alpha)]^{1/2}[W(\alpha)]^{-1/2}}{\kappa+k\beta_2}, \tag{A43}$$

$$l_{22}=\frac{-\kappa[V(\alpha)]^{1/2}[W(\alpha)]^{1/2}(k+\alpha)^{1/2}}{(\kappa+k\beta_1)}+\frac{\kappa[V(\alpha)]^{1/2}[W(\alpha)]^{-1/2}(k+\alpha)^{1/2}}{(\kappa+k\beta_2)}. \tag{A44}$$

Asymptotic growth estimates

From the expressions (A37) to (A44), and the results (B15), (B17) of Appendix B, we obtain the following growth estimates for large $|\alpha|$,

$$\begin{aligned} u_{11}(\alpha)&=O(1), & u_{12}(\alpha)&=O(\alpha^{1/2}),\\ u_{21}(\alpha)&=O(1), & u_{22}(\alpha)&=O(\alpha^{1/2}), \end{aligned} \tag{A45}$$

Det $\boldsymbol{U}(\alpha)=O(\alpha^{1/2})$, as $|\alpha|\to\infty$ in $|\arg(k+\alpha)|<\pi$;

$$\begin{aligned} l_{11}(\alpha)&=O(\alpha^{-1}), & l_{12}(\alpha)&=O(\alpha^{-1/2}),\\ l_{21}(\alpha)&=O(1), & l_{22}(\alpha)&=O(\alpha^{1/2}), \end{aligned} \tag{A46}$$

Det $\boldsymbol{L}(\alpha)=O(\alpha^{-1/2})$, as $|\alpha|\to\infty$ in $|\arg(k+\alpha)|<\pi$. In the situation when $\alpha\to -k$

the expressions (A37) to (A44) in conjunction with the results (B18) and (B19) of Appendix B give

$$\begin{aligned} &u_{11}(\alpha) = O\big((k+\alpha)^{-1/2}\big), \qquad u_{12}(\alpha) = O(1),\\ &u_{21}(\alpha) = O\big((k+\alpha)^{-1/2}\big), \qquad u_{22}(\alpha) = O(1), \end{aligned} \tag{A47}$$

$$\operatorname{Det} \boldsymbol{U}(\alpha) = O\big((k+\alpha)^{-1/2}\big);$$

$$\begin{aligned} &l_{11}(\alpha) = O\big((k+\alpha)^{-1/2}\big), \qquad l_{12}(\alpha) = O(1),\\ &l_{21}(\alpha) = O(1), \qquad l_{22}(\alpha) = O\big((k+\alpha)^{1/2}\big), \end{aligned} \tag{A48}$$

$$\operatorname{Det} \boldsymbol{L}(\alpha) = O(1);$$

$$\text{as} \quad \alpha \to -k, \qquad \operatorname{Re}\beta_1 > 0, \qquad \operatorname{Re}\beta_2 > 0.$$

Appendix B

In this appendix we shall give explicit expressions for integrals which appear in Appendix A, namely

$$I(\alpha) = \frac{-1}{2\pi}\int_{-\infty}^{-k} \frac{1}{|k+t|^{1/2}} \log\left|\frac{|\kappa|^2 + k^2\beta_1^2}{|\kappa|^2 + k^2\beta_2^2}\right| \frac{\mathrm{d}t}{t-\alpha}, \tag{B1}$$

and

$$J(\alpha) = \frac{1}{2\pi i}\int_{-\infty}^{-k} \log\left[\left(\frac{|\kappa| - ik\beta_1}{|\kappa| + ik\beta_1}\right)\left(\frac{|\kappa| - ik\beta_2}{|\kappa| + ik\beta_2}\right)\right] \frac{\mathrm{d}t}{t-\alpha} \tag{B2}$$

where $|\kappa| = \sqrt{t^2 - k^2}$ for $-\infty < t < -k$.

Evaluation of $I(\alpha)$

$I(\alpha)$ can be written as

$$\begin{aligned} I(\alpha) &= \frac{1}{2\pi}\int_k^\infty \frac{\{\log[t^2 - k^2(1-\beta_1^2)] - \log[t^2 - k^2(1-\beta_2^2)]\}\mathrm{d}t}{(t-k)^{1/2}(t+\alpha)}\\ &= \frac{1}{2\pi}\int_k^\infty \Big\{\log\big[t + k(1-\beta_1^2)^{1/2}\big] + \log\big[t - k(1-\beta_1^2)^{1/2}\big]\\ &\quad - \log\big[t + k(1-\beta_2^2)^{1/2}\big] - \log\big[t - k(1-\beta_2^2)^{1/2}\big]\Big\} \frac{\mathrm{d}t}{(t-k)^{1/2}(t+\alpha)} \end{aligned}$$

$$= \frac{1}{2\pi} \int_0^{\infty} \{ \log[t + kB_1(+)] + \log[t + kB_1(-)]$$

$$- \log[t + kB_2(+)] - \log[t + kB_2(-)] \} \frac{dt}{t^{1/2}(t + k + \alpha)}$$

where $B_{1,2}(\pm) = 1 \pm \sqrt{1 - \beta_{1,2}^2}$.

We now use the result [12, form. 14.2(27)]

$$\int_0^{\infty} \frac{\log(t + \delta)}{t^{1/2}(t + \gamma)} dt = \frac{2\pi}{\sqrt{\gamma}} \log(\sqrt{\gamma} + \sqrt{\delta}),$$

$$|\arg \gamma| < \pi, \qquad |\arg \delta| < \pi;$$

giving

$$I(\alpha) = (k + \alpha)^{-1/2} \log \left[\frac{\left(\sqrt{k + \alpha} + \sqrt{kB_1(+)}\right)\left(\sqrt{k + \alpha} + \sqrt{kB_1(-)}\right)}{\left(\sqrt{k + \alpha} + \sqrt{kB_2(+)}\right)\left(\sqrt{k + \alpha} + \sqrt{kB_2(-)}\right)} \right], \tag{B3}$$

$$|\arg(k + \alpha)| < \pi, \qquad \operatorname{Re} \beta_1 > 0, \qquad \operatorname{Re} \beta_2 > 0.$$

Evaluation of $J(\alpha)$

$J(\alpha)$ given by the expression (B2) can be written as

$$J(\alpha) = \int_{\infty}^{\alpha} \left(\frac{1}{2\pi i} \int_{-\infty}^{-k} \log \left[\left(\frac{|\kappa| - ik\beta_1}{|\kappa| + ik\beta_1} \right) \left(\frac{|\kappa| - ik\beta_2}{|\kappa| + ik\beta_2} \right) \right] \frac{dt}{(t - u)^2} \right) du \tag{B4}$$

$$= J_1(\alpha) + J_2(\alpha), \tag{B5}$$

where

$$J_1(\alpha) = \int_{\infty}^{\alpha} \left(\frac{1}{2\pi i} \int_{-\infty}^{-k} \log \left(\frac{|\kappa| - ik\beta_1}{|\kappa| + ik\beta_1} \right) \frac{dt}{(t - u)^2} \right) du, \tag{B6}$$

$$J_2(\alpha) = \int_{\infty}^{\alpha} \left(\frac{1}{2\pi i} \int_{-\infty}^{-k} \log \left(\frac{|\kappa| - ik\beta_2}{|\kappa| + ik\beta_2} \right) \frac{dt}{(t - u)^2} du \right). \tag{B7}$$

We shall now evaluate $J_1(\alpha)$. By carrying out an integration by parts, $J_1(\alpha)$ can be written as

$$J_1(\alpha) = \int_{\infty}^{\alpha} \left[-\frac{1}{2(u + k)} + \frac{1}{2\pi i} \int_{-\infty}^{-k} \frac{d}{dt} \left\{ \log \left(\frac{|\kappa| - ik\beta_1}{|\kappa| + ik\beta_1} \right) \right\} \frac{dt}{t - u} \right] du$$

$$= \int_{\infty}^{\alpha} \left[-\frac{1}{2(u + k)} + Q_1(u) \right] du, \tag{B8}$$

where

$$Q_1(u) = \frac{1}{2\pi i}\int_k^\infty \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\log\left(\frac{|\kappa| - ik\beta_1}{|\kappa| + ik\beta_1}\right)\right)}{t+u}\mathrm{d}t$$

$$= \frac{1}{2\pi i}\int_k^\infty \left(\frac{t}{|\kappa|(|\kappa| - ik\beta_1)} - \frac{t}{|\kappa|(|\kappa| + ik\beta_1)}\right)\frac{\mathrm{d}t}{t+u}$$

$$= \frac{k\beta_1}{\pi}\int_k^\infty \frac{t\mathrm{d}t}{\sqrt{t^2 - k^2}\left\{t^2 - k^2(1\text{-}\beta_1^2)\right\}(t+u)}$$

$$= \frac{k\beta_1}{\pi}\int_k^\infty \frac{t\mathrm{d}t}{\sqrt{t^2 - k^2}\left(t - k\sqrt{1-\beta_1^2}\right)\left(t + k\sqrt{1-\beta_1^2}\right)(t+u)}.$$

Expanding the integrand of the last expression into partial fractions gives

$$Q_1(u) = \frac{k\beta_1}{2\pi}\int_k^\infty \left(\frac{1}{t - k\sqrt{1-\beta_1^2}} + \frac{1}{t + k\sqrt{1-\beta_1^2}}\right)\frac{\mathrm{d}t}{(t+u)\sqrt{t^2-k^2}}$$

$$= \frac{k\beta_1}{2\pi}\int_k^\infty \left(\frac{1}{u + k\sqrt{1-\beta_1^2}}\left(\frac{1}{t - k\sqrt{1-\beta_1^2}} - \frac{1}{t+u}\right)\right.$$

$$\left. + \frac{1}{u - k\sqrt{1-\beta_1^2}}\left(\frac{1}{t + k\sqrt{1-\beta_1^2}} - \frac{1}{t+u}\right)\right)\frac{\mathrm{d}t}{\sqrt{t^2-k^2}}. \tag{B9}$$

We now use the result

$$\int_k^\infty \frac{\mathrm{d}t}{\sqrt{t^2-k^2}(t+\delta)} = \frac{\cos^{-1}(\delta/k)}{\sqrt{k^2-\delta^2}}, \qquad |\arg(k+\delta)| < \pi, \qquad \cos^{-1}(0) = \frac{\pi}{2},$$

in the expression (B9), giving

$$Q_1(u) = \frac{k\beta_1}{2\pi\left(u + k\sqrt{1-\beta_1^2}\right)}\left(\frac{\cos^{-1}\left(-\sqrt{1-\beta_1^2}\right)}{\sqrt{k^2 - k^2(1-\beta_1^2)}} - \frac{\cos^{-1}(u/k)}{\sqrt{k^2-u^2}}\right)$$

$$+ \frac{k\beta_1}{2\pi\left(u - k\sqrt{1-\beta_1^2}\right)}\left(\frac{\cos^{-1}\left(\sqrt{1-\beta_1^2}\right)}{\sqrt{k^2 - k^2(1-\beta_1^2)}} - \frac{\cos^{-1}(u/k)}{\sqrt{k^2-u^2}}\right), \tag{B10}$$

$$0 < \mathrm{Re}\cos^{-1}(u/k) \leqslant \pi, \qquad \mathrm{Re}\left(\sqrt{k^2-u^2}\right) \geqslant 0,$$

$$\mathrm{Re}(\beta_1) > 0, \qquad |\arg(k+u)| < \pi.$$

In (B10) both the functions $\cos^{-1}(u/k)$ and $\sqrt{k^2-u^2}$ have branch cuts at $-\infty < u \leqslant -k$ and $k \leqslant u < \infty$. The quotient $\cos^{-1}(u/k)/\sqrt{k^2-u^2}$ is however continuous across $k \leqslant u < \infty$, hence, this branch cut can be omitted. Then $Q_1(u)$ is indeed analytic in $|\arg(k+u)| < \pi$ with a single branch cut $-\infty < u \leqslant -k$. We also note that $Q_1(u)$ is analytic at $u = \pm k\sqrt{1-\beta_1^2}$, since the singularities cancel. Substituting (B10) into (B8) gives

$$J_1(\alpha) = \int_\infty^\alpha \left[\frac{-1}{2(u+k)} + \frac{k\beta_1}{2\pi\left(u+k\sqrt{1-\beta_1^2}\right)} \left(\frac{\cos^{-1}\left(-\sqrt{1-\beta_1^2}\right)}{k\beta_1} - \frac{\cos^{-1}(u/k)}{\sqrt{k^2-u^2}} \right) \right.$$
$$\left. + \frac{k\beta_1}{2\pi\left(u-k\sqrt{1-\beta_1^2}\right)} \left(\frac{\cos^{-1}\left(\sqrt{1-\beta_1^2}\right)}{k\beta_1} - \frac{\cos^{-1}(u/k)}{\sqrt{k^2-u^2}} \right) \right] \mathrm{d}u. \qquad \text{(B11)}$$

From the expressions (B6) and (B7) it can be seen that the corresponding result for $J_2(\alpha)$ can be obtained from (B11) by replacing the subscript 1 by 2. Thus

$$J_2(\alpha) = \int_\infty^\alpha \left[\frac{-1}{2(u+k)} + \frac{k\beta_2}{2\pi\left(u+k\sqrt{1-\beta_2^2}\right)} \left(\frac{\cos^{-1}\left(-\sqrt{1-\beta_2^2}\right)}{k\beta_2} - \frac{\cos^{-1}(u/k)}{\sqrt{k^2-u^2}} \right) \right.$$
$$\left. + \frac{k\beta_2}{2\pi\left(u-k\sqrt{1-\beta_2^2}\right)} \left(\frac{\cos^{-1}\left(\sqrt{1-\beta_2^2}\right)}{k\beta_2} - \frac{\cos^{-1}(u/k)}{\sqrt{k^2-u^2}} \right) \right] \mathrm{d}u. \qquad \text{(B12)}$$

Combining (B11) and (B12) into (B5) gives

$$J(\alpha) = \int_\infty^\alpha Q(u)\mathrm{d}u,$$

where

$$Q(u) = \frac{-1}{u+k} + \frac{\cos^{-1}\left(-\sqrt{1-\beta_1^2}\right)}{2\pi\left(u+k\sqrt{1-\beta_1^2}\right)} + \frac{\cos^{-1}\left(\sqrt{1-\beta_1^2}\right)}{2\pi\left(u-k\sqrt{1-\beta_1^2}\right)}$$
$$+ \frac{\cos^{-1}\left(-\sqrt{1-\beta_2^2}\right)}{2\pi\left(u+k\sqrt{1-\beta_2^2}\right)} + \frac{\cos^{-1}\left(\sqrt{1-\beta_2^2}\right)}{2\pi\left(u-k\sqrt{1-\beta_2^2}\right)}$$
$$- \frac{k\beta_1}{2\pi} \frac{\cos^{-1}(u/k)}{\sqrt{k^2-u^2}} \left(\frac{1}{u+k\sqrt{1-\beta_1^2}} + \frac{1}{u-k\sqrt{1-\beta_1^2}} \right)$$
$$- \frac{k\beta_2}{2\pi} \frac{\cos^{-1}(u/k)}{\sqrt{k^2-u^2}} \left(\frac{1}{u+k\sqrt{1-\beta_2^2}} + \frac{1}{u-k\sqrt{1-\beta_2^2}} \right). \qquad \text{(B13)}$$

We note in particular that

$$V(\alpha)=\exp[J(\alpha)]=\frac{\left\{\alpha+k\sqrt{1-\beta_1^2}\right\}^{1/2}\left\{\alpha+k\sqrt{1-\beta_2^2}\right\}^{1/2}}{\alpha+k}$$

$$\times\left[\frac{\alpha-k\sqrt{1-\beta_1^2}}{\alpha+k\sqrt{1-\beta_1^2}}\right]^{\cos^{-1}(\sqrt{1-\beta_1^2})/2\pi}\left[\frac{\alpha-k\sqrt{1-\beta_2^2}}{\alpha+k\sqrt{1-\beta_2^2}}\right]^{\cos^{-1}(\sqrt{1-\beta_2^2})/2\pi}$$

$$\times\exp\left[-\frac{k\beta_1}{2\pi}\int_{\infty}^{\alpha}\frac{\cos^{-1}(u/k)}{\sqrt{k^2-u^2}}\left(\frac{1}{u+k\sqrt{1-\beta_1^2}}+\frac{1}{u-k\sqrt{1-\beta_1^2}}\right)\mathrm{d}u\right]$$

$$\times\exp\left[-\frac{k\beta_2}{2\pi}\int_{\infty}^{\alpha}\frac{\cos^{-1}(u/k)}{\sqrt{k^2-u^2}}\left(\frac{1}{u+k\sqrt{1-\beta_2^2}}+\frac{1}{u-k\sqrt{1-\beta_2^2}}\right)\mathrm{d}u\right] \quad \text{(B14)}$$

$$=0(1),\qquad \text{as } |\alpha|\to\infty,\qquad \operatorname{Re}(\beta_{1,2})>0,\qquad |\arg(k+\alpha)|<\pi. \quad \text{(B15)}$$

Also that

$$W(\alpha)=\exp\left[(k+\alpha)^{1/2}I(\alpha)\right]$$

$$=\frac{\left(\sqrt{k+\alpha}+\sqrt{kB_1(+)}\right)\left(\sqrt{k+\alpha}+\sqrt{kB_1(-)}\right)}{\left(\sqrt{k+\alpha}+\sqrt{kB_2(+)}\right)\left(\sqrt{k+\alpha}+\sqrt{kB_2(-)}\right)} \quad \text{(B16)}$$

$$=O(1)\quad \text{as}\quad |\alpha|\to\infty,\qquad \operatorname{Re}(\beta_{1,2})>0,\qquad |\arg(k+\alpha)|<\pi. \quad \text{(B17)}$$

Furthermore as $\alpha\to -k$,

$$W(\alpha)=O(1),\qquad \operatorname{Re}(\beta_{1,2})>0, \quad \text{(B18)}$$

$$V(\alpha)=O\left((k+\alpha)^{-1}\right),\qquad \operatorname{Re}(\beta_{1,2})>0. \quad \text{(B19)}$$

The result (B19) follows from (see [11, §16])

$$\frac{1}{2\pi i}\int_{-\infty}^{-k}\log\left[\left(\frac{|\kappa|-ik\beta_1}{|\kappa|+ik\beta_1}\right)\left(\frac{|\kappa|-ik\beta_2}{|\kappa|+ik\beta_2}\right)\right]\frac{\mathrm{d}t}{t-\alpha}$$

$$=-\log(k+\alpha)+\text{bounded function, as } \alpha\to -k.$$

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